Differentiated Duopoly Revisited

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Abstract

The present paper explores what happens in the analytical results of a duopoly model with product differentiation when heterogeneity of production cost is introduced. It is shown that there is a possibility that the price strategy is dominant over the quantity strategy even if goods are substitutes.

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1　Introduction

This paper considers product-differentiated duopoly with distinct production costs. Our aim is to show the effects of introducing heterogeneity (i.e., difference) in production cost on the results of differentiated duopoly competition.

In the classical duopoly theory, research interest was concentrated on competitions between firms which are homogeneous in the sense that they produce homogeneous goods with the same production cost, using the same strategy i.e., quantity or price setting. Among considerable efforts devoted to elaborate the classical model there are two important approaches; one is to introduce product differentiation, and the other is to consider a mixture of different strategies in a market. Following both research agendas, it has been demonstrated that a dominant strategy of each firm is to set quantity (price) in a differentiated duopoly if goods are substitutes (complements) [e.g., see Singh/Vives (1984), Cheng (1985), and Okuguchi (1987).] Although the conclusion is clear, it is derived under circumstances in which production costs are the same.

So far, not much has been revealed with respect to introducing heterogeneity in production cost. Each firm's production cost may naturally differ if products are differentiated. This paper shows a possibility that the price strategy is dominant if firms are heterogeneous in cost and goods are substitutes. The qualitatively same results are derived in Matsumoto/Onozaki (2005) with a nonlinear differentiated duopoly model, whereas the main purpose of the present paper is to show that nonlinearity is not obliged to get the results.

2 Model

In this section, we present a differentiated duopoly model in which two types of strategy (i.e., quantity or price setting) are taken by each firm.

2.1 Demand Functions

Two distinct firms exist in a market: one is firm X that produces good x with a constant unit production cost a and sells goods at price P_x, and the other is firm Y that produces good y with a constant unit production cost b and sells goods at price P_y. The goods are differentiated, so that each firm faces different demand functions. Bowley-type inverse demand functions with product differentiation are given by

$$P_x = \alpha_x - \beta_x x - \gamma y\ ,$$
$$P_y = \alpha_y - \gamma x - \beta_y y\ .$$

As we concentrate on heterogeneity in production costs, the following assumption is posited:

Assumption 1 $\alpha_x = \alpha_y = \alpha$ *and* $\beta_x = \beta_y = \beta$.

Dividing by β the both sides of each inverse demand equation under Assumption 1 yields

$$\begin{aligned} p_x &= p_M - x - \theta y, \\ p_y &= p_M - \theta x - y, \end{aligned} \tag{1}$$

where $P_x/\beta = p_x$, $P_y/\beta = p_y$, $\alpha/\beta = p_M$ and $\gamma/\beta = \theta$. The intercept p_M denotes the maximum price to be attained and are assumed as

Assumption 2 $p_M > \max(a, b)$,

because otherwise at least one firm do not continue its operation. The parameter θ denotes the degree of product differentiation. If $\theta = 1$, Eqs.

(1) are reduced to a set of demand functions for homogenous goods (i.e., perfect substitutes). If $\theta = 0$, it means the case of independent products in which the duopoly market turns to be two separate monopoly markets. A positive (negative) θ implies that goods are substitutes (complements). As we get, due to the duality, complementarity from substitutability only by changing the sign of θ, we confine our analysis to the case in which goods are substitutes in this study. Hence we assume

Assumption 3 $0 < \theta < 1$.

For later convenience, by solving (1) simultaneously with respect to x and y, we get direct demands,

$$x = \frac{1}{1-\theta^2}((1-\theta)p_M - p_x + \theta p_y),$$
$$y = \frac{1}{1-\theta^2}((1-\theta)p_M + \theta p_x - p_y). \tag{2}$$

Furthermore, by solving each equation of (1) for x and y respectively, we obtain

$$x = p_M - p_x - \theta y,$$
$$y = p_M - \theta x - p_y. \tag{3}$$

2.2 Four Types of Duopoly Competition and the Corresponding Reaction Functions

Each firm chooses one of the two strategies (i.e., quantity or price) and maximizes its profits, taking the competitor's strategic variable as given. With two firms and two strategies, there are four possible types of duopoly competition according as which firm takes which strategy. In *Cournot-Cournot* (henceforth *CC*) competition, each firm sets quantity, taking its competitor's output as given, and in *Bertrand-Bertrand* (*BB*) competition, each firms sets prices, taking its competitor's prices as given. In *Cournot-*

Bertrand (*CB*) competition, firm X sets quantity, taking p_y as given whereas firm Y sets prices, taking x as given. In *Bertrand-Cournot* (*BC*) competition in which strategies are interchanged, firm X sets prices, taking y as given whereas firm Y sets quantity, taking p_x as given. We call the former two types of competitions as *homogeneous* because both firms follow the same strategy. On the other hand, we call the latter two types as *heterogeneous* because the firms take different strategies.

As we assume firms' marginal cost a and b to be constant, we obtain profit functions as

$$\begin{aligned} \pi_x &= (p_x - a)x, \\ \pi_y &= (p_y - b)y. \end{aligned} \tag{4}$$

In *CC* competition a strategic variable for both firms is quantity, so that we rewrite profit functions (4) by substituting (1) and solve the first-order conditions for profit maximization problem, yielding *CC* reaction functions

$$\begin{aligned} 2x + \theta y &= p_M - a \qquad \text{for firm } X, \\ \theta x + 2y &= p_M - b \qquad \text{for firm } Y. \end{aligned} \tag{5}$$

In *BB* competition a strategic variable for both firm is price, so that substituting (2) yields *BB* reaction functions

$$\begin{aligned} 2p_x - \theta p_y &= (1-\theta)p_M + a \qquad \text{for firm } X, \\ \theta p_x - 2p_y &= (1-\theta)p_M + b \qquad \text{for firm } Y. \end{aligned} \tag{6}$$

In *CB* competition a strategic variable of firm X is quantity and that of firm Y is price, so that substituting the first equation of (1) and the second equation of (3) yields *CB* reaction functions

$$\begin{aligned} 2(1-\theta^2)x-\theta p_y &= (1-\theta)p_M-a \qquad \text{for firm } X, \\ \theta x+2p_y &= p_M+b \qquad \text{for firm } Y. \end{aligned} \tag{7}$$

In BC competition a strategic variable of firm X is price and that of firm Y is quantity, so that substituting the second equation of (1) and the first equation of (3) yields BC reaction functions

$$\begin{aligned} 2p_x+\theta y &= p_M+a \qquad \text{for firm } X, \\ \theta p_x-2(1-\theta^2)y &= -(1-\theta)p_M+b \qquad \text{for firm } Y. \end{aligned} \tag{8}$$

The resultant reaction functions exhibit the symmetrical property to CB competition, i.e., replacing x with y, p_x with p_y, and a with b in (7) yields (8).

3 Analysis of Duopoly Competitions

In this section, we analyze equilibria of four types of duopoly one by one. In the succeeding two subsections, we concentrate on the homogeneous duopoly competitions, which lead us to common-sense results. In the later two subsections, we focus on the heterogeneous duopoly competitions, which lead us beyond common-sense results.

3.1 *CC* Duopoly

All the reaction functions (5)−(8) are linear with respect to strategic variables and a pair of reaction functions in each type of duopoly has only one intersection. The intersection of (5) gives CC equilibrium where outputs are[1]

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[1] Henceforth, superscript *CC, BB, CB* and *BC* is attached to a variable so as to denote that it is an equilibrium value of the corresponding type of duopoly.

$$x^{CC}=\frac{2(p_M-a)-\theta(p_M-b)}{4-\theta^2},$$
$$y^{CC}=\frac{2(p_M-b)-\theta(p_M-a)}{4-\theta^2}, \tag{9}$$

and prices are

$$p_x^{CC}=\frac{p_M(2-\theta)+a(2-\theta^2)+b\theta}{4-\theta^2}>0,$$
$$p_y^{CC}=\frac{p_M(2-\theta)+b(2-\theta^2)+a\theta}{4-\theta^2}>0. \tag{10}$$

Substituting these into (4) yields CC profits,

$$\pi_x^{CC}=(x^{CC})^2>0,$$
$$\pi_y^{CC}=(y^{CC})^2>0. \tag{11}$$

The variables x^{CC} and y^{CC} may take negative value depending on the relative magnitude of a and b. To see this, we assume

Assumption 4 $b-a=k\neq 0$.

Furthermore, let us here say that a firm is *efficient* (*inefficient*) if its marginal cost is lower (higher). Then Assumption 4 implies that firm X is efficient if $k>0$ and firm Y is efficient if $k<0$. Under Assumption 4, Eqs. (9) are rewritten as

$$x^{CC}=\frac{(2-\theta)(p_M-a)+\theta k}{4-\theta^2}>0 \quad \text{if} \quad k>0,$$
$$y^{CC}=\frac{(2-\theta)(p_M-a)-2k}{4-\theta^2}>0 \quad \text{if} \quad k<0.$$

These inequalities imply that CC output may be negative when a firm is inefficient. Solving $x^{CC}=0$ and $y^{CC}=0$ for k yields

$$k_x^{CC} = -\frac{(2-\theta)(p_M - a)}{\theta},$$
$$k_y^{CC} = \frac{(2-\theta)(p_M - a)}{2}$$

Without loss of generality, we may assume

Assumption 5 $p_M - a = 1$.

Thus we obtain the zero-output loci,

$$\begin{aligned} k_{x=0}^{CC}(\theta) &= -\frac{2-\theta}{\theta}, \\ k_{y=0}^{CC}(\theta) &= \frac{2-\theta}{2}, \end{aligned} \tag{12}$$

where $\lim_{\theta \to 0} k_{x=0}^{CC}(\theta) = -\infty$, $k_{x=0}^{CC}(1) = -1$, $k_{x=0}^{CC\prime}(\theta) > 0$. Assumptions 2, 4 and 5 imply $k < 1$, so that in what follows we concentrate on the parameter region $\{(\theta, k) \mid (\theta, k) \in (0,1) \times (-\infty, 1)\}$ These loci do not intersect within this region, therefore it is divided into three subregions by them; two in which either x^{CC} or y^{CC} is negative (the shaded areas in Figure 1) and one in which both are positive. The last one is the parameter region to be considered in what follows.

The differences and the ratio of equilibrium values and their order of magnitude are

$$x^{CC} - y^{CC} = \frac{b-a}{2-\theta} \lesseqgtr 0 \qquad \text{according as} \quad a \gtreqless b,$$
$$p_x^{CC} - p_y^{CC} = \frac{(a-b)(1-\theta)}{2-\theta} \gtreqless 0 \qquad \text{according as} \quad a \gtreqless b,$$
$$\frac{\pi_x^{CC}}{\pi_y^{CC}} = \left(\frac{x^{CC}}{y^{CC}}\right)^2 \lesseqgtr 1 \qquad \text{according as} \quad a \gtreqless b.$$

All the equalities, $x^{CC} = y^{CC}$, $p_x^{CC} = p_y^{CC}$ and $\pi_x^{CC} = \pi_y^{CC}$ imply $k = 0$, then the parameter region to be considered is now divided into two subregions, labelled as CC_1 and CC_2, by $k = 0$ as depicted in Figure 1. The order of magnitude of each firm's *CC* variables in each region is as follows:

$$CC_1=\{(\theta, k)\mid x^{CC}>y^{CC},\ p_x^{CC}<p_y^{CC},\ \pi_x^{CC}>\pi_y^{CC}\},$$
$$CC_2=\{(\theta, k)\mid x^{CC}<y^{CC},\ p_x^{CC}>p_y^{CC},\ \pi_x^{CC}<\pi_y^{CC}\}.$$

Thus the results of comparison in *CC* duopoly are summarized as follows:
Proposition 1 *In CC duopoly, (i) if $a=b$, then $x^{CC}=y^{CC}$, $p_x^{CC}=p_y^{CC}$ and $\pi_x^{CC}=\pi_y^{CC}$, and (ii) if $a\neq b$, then an efficient firm produces more output, faces lower prices and earns more profits.*

3.2 *BB* Duopoly

The intersection of (6) gives *BB* equilibrium where outputs are

$$x^{BB}=\frac{(2-\theta^2)(p_M-a)-\theta(p_M-b)}{(1-\theta^2)(4-\theta^2)},\qquad y^{BB}=\frac{(2-\theta^2)(p_M-b)-\theta(p_M-a)}{(1-\theta^2)(4-\theta^2)},\tag{13}$$

and prices are

$$p_x^{BB}=\frac{p_M(2-\theta-\theta^2)+2a+b\theta}{4-\theta^2}>0,\qquad p_y^{BB}=\frac{p_M(2-\theta-\theta^2)+2b+a\theta}{4-\theta^2}>0.\tag{14}$$

Substituting these into (4) yields *BB* profits,

$$\pi_x^{BB}=(1-\theta^2)(x^{BB})^2>0,\qquad \pi_y^{BB}=(1-\theta^2)(y^{BB})^2>0.\tag{15}$$

The variables x^{BB} and y^{BB} may take negative value depending on if a firm is inefficient. Solving $x^{BB}=0$ and $y^{BB}=0$ for k under Assumptions 4 and 5 yields the zero-output loci,

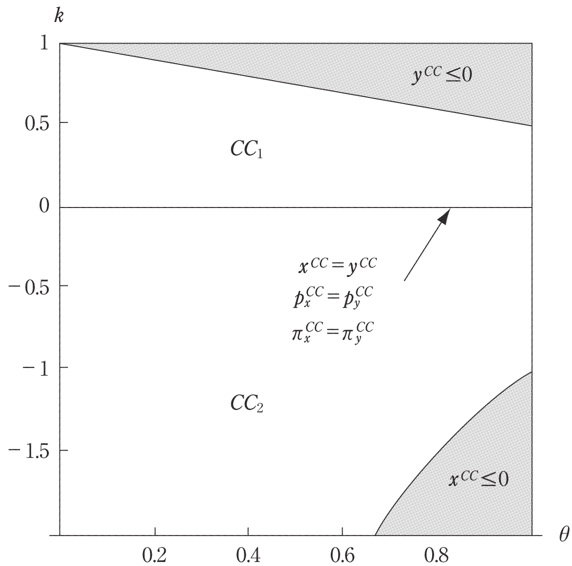

Figure 1: Regime classification of *CC* competitions

$$k_{x=0}^{BB}(\theta) = -\frac{(2+\theta)(1-\theta)}{\theta}, \qquad k_{y=0}^{BB}(\theta) = \frac{(2+\theta)(1-\theta)}{2-\theta^2}, \tag{16}$$

where $\lim_{\theta\to 0} k_{x=0}^{BB}(\theta) = -\infty$, $k_{x=0}^{BB}(1) = 0$, $k_{x=0}^{BB\prime}(\theta) > 0$, $k_{y=0}^{BB}(0) = 1$, $k_{y=0}^{BB}(1) = 0$ and $k_{y=0}^{BB\prime}(\theta) < 0$. The parameter region to be considered in what follows is shown as unshaded area in Figure 2.

The differences and the ratio of equilibrium values and their order of magnitude are

$$x^{BB} - y^{BB} = \frac{b-a}{(1-\theta)(2+\theta)} \lesseqgtr 0 \qquad \text{according as} \quad a \gtreqless b,$$

$$p_x^{BB} - p_y^{BB} = \frac{(a-b)}{2+\theta} \gtreqless 0 \qquad \text{according as} \quad a \gtreqless b,$$

$$\frac{\pi_x^{BB}}{\pi_y^{BB}} = \left(\frac{x^{BB}}{y^{BB}}\right)^2 \lesseqgtr 1 \qquad \text{according as} \quad a \gtreqless b.$$

All the equalities, $x^{BB}=y^{BB}$, $p_x^{BB}=p_y^{BB}$ and $\pi_x^{BB}=\pi_y^{BB}$ again imply $k=0$, then the parameter region to be considered is divided into two subregions, labelled as BB_1 and BB_2, by $k=0$ as depicted in Figure 2. The order of magnitude of each firm's BB variables in each region is as follows:

$$BB_1 = \{(\theta, k) \mid x^{BB} > y^{BB},\ p_x^{BB} < p_y^{BB},\ \pi_x^{BB} > \pi_y^{BB}\},$$
$$BB_2 = \{(\theta, k) \mid x^{BB} < y^{BB},\ p_x^{BB} > p_y^{BB},\ \pi_x^{BB} < \pi_y^{BB}\}.$$

Thus the results of comparison in BB duopoly, which is just the same as that of CC duopoly, are summarized as follows:

Proposition 2 *In BB-duopoly,* (*i*) *if* $a=b$, *then* $x^{BB}=y^{BB}$, $p_x^{BB}=p_y^{BB}$ *and* $\pi_x^{BB}=\pi_y^{BB}$, *and* (*ii*) *if* $a \neq b$, *then an efficient firm produces more output, faces*

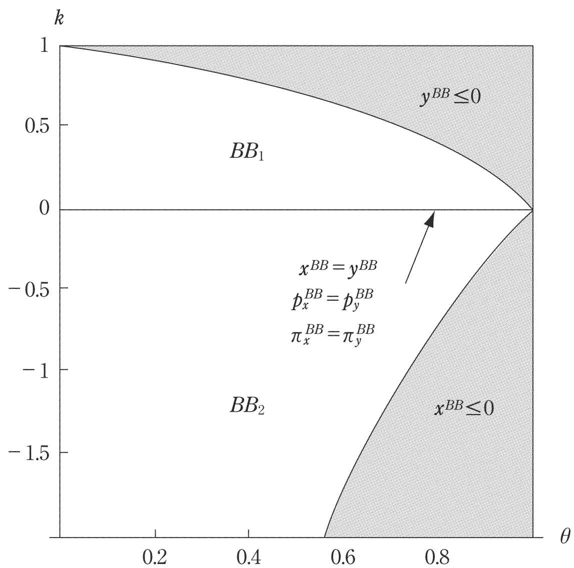

Figure 2: Regime classification of BB competitions

lower prices and earns more profits.

As shown above, consideration of homogeneous duopoly competitions brings us to common-sense results whether with or without cost difference. Next, we concentrate on heterogeneous duopoly competitions, which lead us beyond common-sense results if there is cost difference.

3.3 *CB* Duopoly

The intersection of (7) gives CB equilibrium where outputs are

$$x^{CB}=\frac{2(p_M-a)-\theta(p_M-b)}{4-3\theta^2},$$
$$y^{CB}=\frac{(2-\theta^2)(p_M-b)-\theta(p_M-a)}{4-3\theta^2}, \tag{17}$$

and prices are

$$p_x^{CB}=\frac{a(2-\theta^2)+(1-\theta^2)(b\theta+p_M(2-\theta-\theta^2))}{4-3\theta^2}>0,$$
$$p_y^{CB}=\frac{a\theta+2b(1-\theta^2)+p_M(2-\theta-\theta^2)}{4-3\theta^2}>0.$$

Substituting these into (4) yields CB profits,

$$\pi_x^{CB}=(1-\theta^2)(x^{CB})^2>0,$$
$$\pi_y^{CB}=(y^{CB})^2>0. \tag{18}$$

The differences of each equilibrium values are

$$x^{CB}-y^{CB}=\frac{\theta^2(p_M-b)+(2+\theta)(b-a)}{4-3\theta^2},$$
$$p_x^{CB}-p_y^{CB}=(1-\theta)(y^{CB}-x^{CB}),$$
$$\pi_x^{CB}-\pi_y^{CB}=\frac{(a-b)(2p_M-a-b)(5\theta^2-4)+2\theta^3(p_M-b)((p_M-a)-\theta(p_M-b))}{(4-3\theta^2)^2}.$$

The second equation implies that the order of magnitude of prices is always opposite of that of output. Thus we first obtain the following results of the homogeneous cost case:

Proposition 3 *In CB duopoly, if* $a=b$, *then* $x^{CB}>y^{CB}$, $p_x^{CB}<p_y^{CB}$ *and* $\pi_x^{CB}>\pi_y^{CB}$; *in other words, a quantity-setter produces more outputs, faces lower prices and earns more profits.*

Next we consider the case where $a\neq b$. In this case, the variables x^{CB} and y^{CB} may take negative value depending on if a firm is inefficient. Solving $x^{CB}=0$ and $y^{CB}=0$ for k under Assumptions 4 and 5 yields the zero-output loci,

$$k_{x=0}^{CB}(\theta)=-\frac{2-\theta}{\theta},$$
$$k_{y=0}^{CB}(\theta)=\frac{(2+\theta)(1-\theta)}{2-\theta^2}, \tag{19}$$

where $\lim_{\theta\to 0}k_{x=0}^{CB}(\theta)=-\infty$, $k_{x=0}^{CB}(1)=-1$, $k_{x=0}^{CB\prime}(\theta)>0$, $k_{y=0}^{CB}(0)=1$, $k_{y=0}^{CB}(1)=0$ and $k_{y=0}^{CB\prime}(\theta)<0$. The parameter region to be considered in what follows is shown as unshaded area in Figure 3.

Solving $x^{CB}=y^{CB}$ under Assumptions 4 and 5 for k yields the equi-output locus,

$$k_{x=y}^{CB}(\theta)=-\frac{\theta^2}{(2-\theta)(1+\theta)}, \tag{20}$$

where $k_{x=y}^{CB}(0)=0$, $k_{x=y}^{CB}(1)=-0.5$ and $k_{x=y}^{CB\prime}(\theta)<0$. Solving $p_x^{CB}=p_y^{CB}$ under Assumptions 4 and 5 yields the equi-price locus,

$$k_{p_x=p_y}^{CB}(\theta)=-\frac{\theta^2}{(2-\theta)(1+\theta)}, \tag{21}$$

which is identical with the equi-output locus (20). Solving $\pi_x^{CB}=\pi_y^{CB}$ under Assumptions 4 and 5 gives the equi-profit locus,

$$k_{\pi_x=\pi_y}^{CB}(\theta)=\frac{\sqrt{1-\theta}\,((2+\theta)\sqrt{1-\theta}-(2-\theta)\sqrt{1+\theta})}{2-\theta^2+\theta\sqrt{1-\theta^2}}, \tag{22}$$

where $k_{\pi_x=\pi_y}^{CB}(0)=0$, $k_{\pi_x=\pi_y}^{CB}(1)=0$ and $k_{\pi_x=\pi_y}^{CB}(\theta)<0$. The last inequality holds because $(2+\theta)^2(1-\theta)-(2-\theta)^2(1+\theta)=-2\theta^3<0$, which implies the negativity of the numerator.

The parameter region to be considered is now divided into three subregions, labelled as CB_1, CB_2, and CB_3, by the equi-output locus (which is identical with the equi-price locus) and the equi-profit locus as depicted in Figure 3. The order of magnitude of each firm's CB variables in each region is as follows:

$$CB_1 = \{(\theta, k) \mid x^{CB} > y^{CB},\ p_x^{CB} < p_y^{CB},\ \pi_x^{CB} > \pi_y^{CB}\},$$
$$CB_2 = \{(\theta, k) \mid x^{CB} > y^{CB},\ p_x^{CB} < p_y^{CB},\ \pi_x^{CB} < \pi_y^{CB}\},$$
$$CB_3 = \{(\theta, k) \mid x^{CB} < y^{CB},\ p_x^{CB} > p_y^{CB},\ \pi_x^{CB} < \pi_y^{CB}\}.$$

In the region CB_1 with $k > 0$, firm X is efficient, produces more output, faces lower prices and makes more profits than firm Y. In the region CB_1 with $k < 0$, firm X is inefficient, still produces more output and makes more profits. In the region CB_2, where $k < 0$, firm X is inefficient, still produces more output but makes less profits. Finally, in the region CB_3, firm X is inefficient, faces lower prices and makes less profits than firm Y. By symmetry of duopoly, these results are reversed from the viewpoint of firm Y. In the region CB_3, for example, firm Y is efficient, produces more output, sets lower prices and makes more profits than firm X. The results of CB competition are summarized as follows:

Proposition 4 *In CB duopoly, if $a \neq b$, then an efficient firm produces more output, faces lower prices and earns more profits regardless of whether it is a quantity-setter or a price-setter. If a price-setter is efficient, it may earn more profits than a quantity-setter even though it produces less output and sets higher prices. Furthermore, a quantity-setter may earns more profits even though it is inefficient.*

3.4 *BC* Duopoly

The intersection of (8) gives BC equilibrium where outputs are

$$x^{BC}=\frac{(2-\theta^2)(p_M-a)-\theta(p_M-b)}{4-3\theta^2},$$
$$y^{BC}=\frac{2(p_M-b)-\theta(p_M-a)}{4-3\theta^2}, \tag{23}$$

and prices are

$$p_x^{BC}=\frac{b\theta+2a(1-\theta^2)+p_M(2-\theta-\theta^2)}{4-3\theta^2}>0,$$
$$p_y^{BC}=\frac{b(2-\theta^2)+(1-\theta^2)(a\theta+p_M(2-\theta-\theta^2))}{4-3\theta^2}>0.$$

Substituting these into (4) yields BC profits,

$$\pi_x^{BC}=(x^{BC})^2>0,$$
$$\pi_y^{BC}=(1-\theta^2)(y^{BC})^2>0. \tag{24}$$

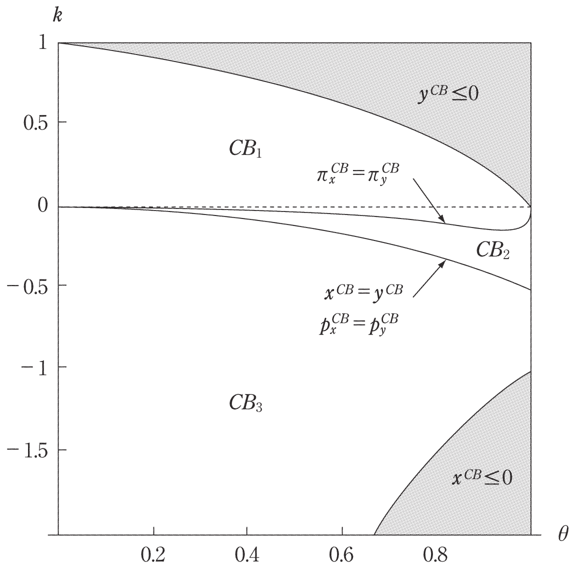

Figure 3: Regime classification of CB competitions

The differences of each equilibrium values are

$$x^{BC}-y^{BC}=-\frac{\theta^2(p_M-a)+(2+\theta)(a-b)}{4-3\theta^2},$$

$$p_x^{BC}-p_y^{BC}=(1-\theta)(y^{BC}-x^{BC}),$$

$$\pi_x^{BC}-\pi_y^{BC}=\frac{(a-b)(2p_M-a-b)(5\theta^2-4)+2\theta^3(p_M-a)((p_M-b)-\theta(p_M-a))}{(4-3\theta^2)^2}.$$

The second equation again implies that the order of magnitude of prices is always opposite of that of output. Thus we first obtain the following results of the homogeneous cost case:

Proposition 5 *In BC duopoly, if* $a=b$*, then* $x^{BC}<y^{BC}$, $p_x^{BC}>p_y^{BC}$ *and* $\pi_x^{BC}<\pi_y^{BC}$*; in other words, a quantity-setter produces more outputs, faces lower prices and earns more profits.*

Next we introduce the heterogeneity of production costs. Under Assumptions 4 and 5, we obtain the zero-output loci,

$$k_{x=0}^{BC}(\theta)=-\frac{(2+\theta)(1-\theta)}{\theta},$$
$$k_{y=0}^{BC}(\theta)=\frac{2-\theta}{2}, \tag{25}$$

where $\lim_{\theta\to 0}k_{x=0}^{BC}(\theta)=-\infty$, $k_{x=0}^{BC}(1)=0$, $k_{x=0}^{BC\prime}(\theta)>0$, $k_{y=0}^{BC}(0)=1$, $k_{y=0}^{BC}(1)=0.5$ and $k_{y=0}'(\theta)<0$. The parameter region to be considered in what follows is shown as unshaded area in Figure 4.

In the same way as the previous subsection, we obtain the equi-output locus,

$$k_{x=y}^{BC}(\theta)=\frac{\theta^2}{2+\theta}, \tag{26}$$

where $k_{x=y}^{BC}(0)=0$, $k_{x=y}^{BC}(1)=\frac{1}{3}$ and $k_{x=y}^{BC\prime}(\theta)>0$. The equi-price locus is given by

$$k_{p_x=p_y}^{BC}(\theta)=\frac{\theta^2}{2+\theta}, \tag{27}$$

which is, again, identical with the equi-output locus (26). Finally, the equiprofit locus is

$$k^{BC}_{\pi x=\pi y}(\theta)=\frac{\sqrt{1-\theta}\,((2-\theta)\sqrt{1+\theta}-(2+\theta)\sqrt{1-\theta})}{\theta+2\sqrt{1-\theta^2}}, \tag{28}$$

where $k^{BC}_{\pi x=\pi y}(0)=0$, $k^{BC}_{\pi x=\pi y}(1)=0$ and $k^{BC}_{\pi x=\pi y}(\theta)>0$.

The parameter region to be considered is, again, divided into three subregions, labelled as BC_1, BC_2 and BC_3, by the equi-output locus (which is identical with the equi-price locus) and the equi-profit locus as depicted in Figure 4. The order of magnitude of each firm's BC variables in each region is as follows:

$$BC_1=\{(\theta,k)\mid x^{BC}>y^{BC},\ p_x^{BC}<p_y^{BC},\ \pi_x^{BC}>\pi_y^{BC}\},$$
$$BC_2=\{(\theta,k)\mid x^{BC}<y^{BC},\ p_x^{BC}>p_y^{BC},\ \pi_x^{BC}>\pi_y^{BC}\},$$
$$BC_3=\{(\theta,k)\mid x^{BC}<y^{BC},\ p_x^{BC}>p_y^{BC},\ \pi_x^{BC}<\pi_y^{BC}\}.$$

In the region BC_1, firm X is efficient, produces more output, sets lower prices and makes more profits than firm Y. In the region BC_2, firm X is still efficient, but produces less output and sets higher prices, therefore earns more profits. In the region BC_3 with $k<0$, firm Y is efficient, produces more output, faces lower prices and earns more profits. In the region BC_3 with $k>0$, firm Y is inefficient, still produces more output and earns more profits. therefore, the results of BC competition are summarized just the same as Proposition 4:

Proposition 6 *In BC duopoly, if $a\neq b$, then an efficient firm produces more output, faces lower prices and earns more profits regardless of whether it is a quantity-setter or a price-setter. If a price-setter is efficient, it may earn more profits than a quantity-setter even though it produces less output and sets higher prices. Furthermore, a quantity-setter may earns more profits even though it is inefficient.*

It is worthwhile to note that Proposition 3 and 5, both applying to the case where $a=b$, reconfirm one of the main results of Singh/Vives (1984), whereas Proposition 4 and 6 extend it to the case where $a\neq b$.

4 Comparison of Equilibria across Different Types of Duopoly Competition

In this section, we concentrate on the effect on profits caused by choice of strategy. We compare profits of a particular firm across different duopoly competitions. Such comparison allows us to understand the impact of strategic behavior on profits in a differentiated duopoly market.

From Eqs. (11), (15), (18) and (24), profit differences of firm X across different competitions are obtained as[2]

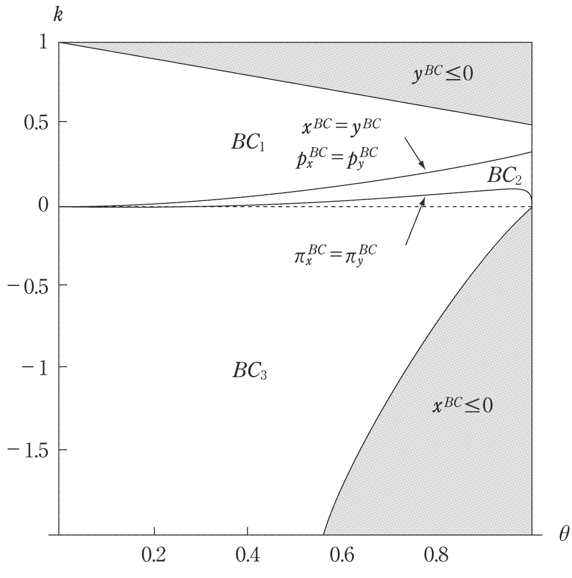

Figure 4: Regime classification of BC competitions

[2] From symmetry, the analogous discussion also follows with respect to firm Y.

$$\pi_x^{CC} - \pi_x^{BB} = \frac{\theta^3 A_1}{(4-\theta^2)^2(1-\theta^2)} \gtrless 0,$$

$$\pi_x^{CC} - \pi_x^{CB} = \frac{\theta^6(2-\theta+k\theta)^2}{(4-3\theta^2)^2(4-\theta^2)^2} > 0,$$

$$\pi_x^{CC} - \pi_x^{BC} = \frac{\theta^3 A_2 A_3}{(4-3\theta^2)^2(4-\theta^2)^2} \gtrless 0,$$

$$\pi_x^{BB} - \pi_x^{CB} = \frac{\theta^3 A_4 A_5}{(4-3\theta^2)^2(4-\theta^2)^2(1-\theta^2)} \gtrless 0,$$

$$\pi_x^{BB} - \pi_x^{BC} = \frac{\theta^6(2-\theta+k\theta-\theta^2)^2}{(4-3\theta^2)^2(4-\theta^2)^2(1-\theta^2)} > 0,$$

$$\pi_x^{CB} - \pi_x^{BC} = \frac{\theta^3 A_1}{(4-3\theta^2)^2} \gtrless 0,$$

where

$$A_1 = 2-2\theta-2k(1-\theta)-k^2\theta,$$
$$A_2 = 2-\theta-2k,$$
$$A_3 = 16-4k(2-\theta^2)-\theta(8+\theta(2-\theta)(6+\theta)),$$
$$A_4 = -2+\theta+(1-k)\theta^2+2k,$$
$$A_5 = 16-8\theta-20\theta^2+8\theta^3+5\theta^4-\theta^5+k\theta(8-8\theta^2+\theta^4).$$

From the second and fifth equations, it follows that

$$\pi_x^{CC} > \pi_x^{CB} \quad \text{and} \quad \pi_x^{BB} > \pi_x^{BC}. \tag{29}$$

Other orders of magnitude between strategy combinations are not directly determined. To complete comparison, we compare the curves, $A_i = 0$, $i = 1, \ldots, 5$, with the zero-output loci in *CB* and *BC* competitions. It should be noted here that, from Eqs. (12), (16), (19) and (25), the zero-output loci $k_{x=0}^{CC}(\theta)=0$ and $k_{x=0}^{CB}(\theta)=0$ are identical, the loci $k_{y=0}^{CC}(\theta)=0$ and $k_{y=0}^{BC}(\theta)=0$ identical, the loci $k_{x=0}^{BB}(\theta)=0$ and $k_{x=0}^{BC}(\theta)=0$ identical, and the loci $k_{y=0}^{BB}(\theta)=0$ and $k_{y=0}^{CB}(\theta)=0$ also identical. Furthermore,

$$\{(\theta,k)\mid x^{BC}\leq 0\}\subset\{(\theta,k)\mid x^{CB}\leq 0\}=\{(\theta,k)\mid x^{CC}\leq 0\}$$

and

$$\{(\theta,k)\mid y^{CB}\leq 0\}\subset\{(\theta,k)\mid y^{BC}\leq 0\}=\{(\theta,k)\mid y^{CC}\leq 0\},$$

thereby we only have to consider the loci $k^{BC}_{x=0}(\theta)=0$ and $k^{CB}_{y=0}(\theta)=0$ as the strictest positivity-condition for output.

Solving $A_1=0$ for k gives

$$k_1(\theta)=\frac{-1+\theta+\sqrt{1-\theta^2}}{\theta}>0$$

and

$$k_2(\theta)=\frac{-1+\theta-\sqrt{1-\theta^2}}{\theta}<0.$$

The difference between $k_1(\theta)$ and $k^{CB}_{y=0}(\theta)$ is

$$k_1(\theta)-k^{CB}_{y=0}(\theta)=\frac{\sqrt{1-\theta^2}\,(2-\theta^2-2\sqrt{1-\theta^2})}{\theta(2-\theta^2)}>0.$$

The inequality follows because $(2-\theta^2)^2>(2\sqrt{1-\theta^2})^2$. The graph of $k_1(\theta)$ lies above that of $k^{CB}_{y=0}(\theta)$ in the (θ,k)-plane, which implies that $\pi^{CC}_x>\pi^{BB}_x$ for $k>0$ if $y^{CB}>0$. The difference between $k_2(\theta)$ and $k^{BC}_{x=0}(\theta)$ is

$$k_2(\theta)-k^{BC}_{x=0}(\theta)=\frac{1-\theta-\sqrt{1-\theta^2}}{\theta}<0.$$

The inequality follows because $(1-\theta)^2<(\sqrt{1-\theta^2})^2$. Thus the graph of $k_2(\theta)$ lies below that of $k^{BC}_{x=0}(\theta)$ in the (θ,k)-plane, which implies that $\pi^{CC}_x>\pi^{BB}_x$ for $k<0$ if $x^{BC}>0$. Here we can confirm the result of a classical duopoly model with homogeneous goods that quantity is lower, price is higher and profits are higher under quantity competition than under price competition, even if marginal costs are different[3].

Solving $A_2=0$ for k yields

[3] Comparison of quantities and prices is straightforward, so it is left to the readers.

$$k_3(\theta)=\frac{2-\theta}{2}>0.$$

The difference between $k_3(\theta)$ and $k^{CB}_{y=0}(\theta)$ is

$$k_3(\theta)-k^{CB}_{y=0}(\theta)=\frac{\theta^3}{4-2\theta^2}>0,$$

which implies the graph of $k_3(\theta)$ lies above that of $k^{CB}_{y=0}(\theta)$ in the (θ, k)-plane. In addition, solving $A_3=0$ for k yields

$$k_4(\theta)=\frac{(2-\theta)(\theta(6+\theta)-8)}{4\theta(2-\theta^2)}<0.$$

The difference between $k_4(\theta)$ and $k^{BC}_{x=0}(\theta)$ is

$$k_4(\theta)-k^{BC}_{x=0}(\theta)=-\frac{4\theta-3\theta^3}{8-4\theta^2}<0,$$

which implies the graph of $k_4(\theta)$ lies below that of $k^{BC}_{x=0}(\theta)$ in the (θ, k)-plane. Both of these imply that $\pi^{CC}_x>\pi^{BC}_x$ if $x^{BC}>0$ and $y^{CB}>0$.

Solving $A_4=0$ for k gives

$$k_5(\theta)=\frac{(1-\theta)(2+\theta)}{2-\theta^2}>0,$$

which is identical to $k^{CB}_{y=0}$. Solving $A_5=0$ for k yields

$$k_6(\theta)=\frac{4(1-\theta^2)(\theta^2+2\theta-4)-\theta^4(1-\theta)}{\theta(8(1-\theta^2)+\theta^4)}<0.$$

The difference between $k_6(\theta)$ and $k^{BC}_{x=0}(\theta)$ is

$$k_6(\theta)-k^{BC}_{x=0}(\theta)=-\frac{\theta(4-\theta^2)(1-\theta^2)}{8(1-\theta^2)+\theta^4}<0,$$

which implies the graph of $k_6(\theta)$ lies below that of $k^{BC}_{x=0}(\theta)$ in the (θ, k)-plane. Both of these facts imply that $\pi^{CB}_x>\pi^{BB}_x$ if $x^{BC}>0$ and $y^{CB}>0$.

From (29) and above consideration, it follows:

Proposition 7 $\pi^{CC}_x>\pi^{CB}_x>\pi^{BB}_x>\pi^{BC}_x$ *if* $x^{BC}>0$ *and* $y^{CB}>0$. *Equivalently,* $\pi^{CC}_y>\pi^{BC}_y>\pi^{BB}_y>\pi^{CB}_y$ *if* $x^{BC}>0$ *and* $y^{CB}>0$.

If we suppose the same two-stage game that Singh/Vives (1984) studies, the above leads us to the same result that choosing a quantity strategy is dominant for both firms and therefore *CC* competition is a Nash equilibrium.

Finally we summarize the result of profit comparison in *CB* and *BC* competitions. As shown in Figure 5, the two equi-profit loci, $\pi_x^{CB}=\pi_y^{CB}$ and $\pi_x^{BC}=\pi_y^{BC}$ divide the region where $x^{BC}>0$ (or $x^{BB}>0$) and $y^{CB}>0$ (or $y^{BB}>0$) into three subregions, which are denoted as I, II and III. From Proposition 7, the order of strategy combinations does not depend upon region. In region I and II, the order of magnitude of profits are as follows:

$$\begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \vee & & \vee \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array} \qquad \begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \wedge & & \wedge \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array}$$

region I　　　　region II

The results imply that the difference in the order of magnitude of profits is due to the difference in production costs. The parameter k is positive in region I, so that firm X is efficient and makes more profits in *CB* and *BC* competitions. The parameter k is negative in region II, so that firm Y is efficient and makes more profits in both competitions. To sum up, an efficient firm makes more profits regardless of which strategy it takes in region I and II. In region III, the ordering of magnitude of profits are as follows:

$$\begin{array}{ccc} \pi_x^{CB} & > & \pi_x^{BC} \\ \vee & & \wedge \\ \pi_y^{CB} & < & \pi_y^{BC} \end{array}$$

region III

This time a quantity-setter makes more profits regardless of whether it is efficient. Thus we obtain the following result:

Proposition 8　*If cost difference is relatively large, an efficient firm makes more profits regardless of its strategy. If cost difference is relatively small, a*

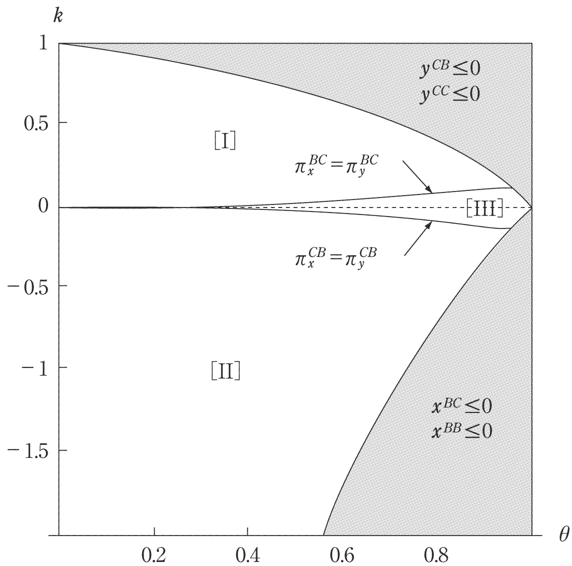

Figure 5: Profit comparison

quantity-setter makes more profits regardless of whether it is efficient.

5 Concluding Remarks

We may summarize the results derived from the model as follows:

(1) In homogeneous duopolies, if production costs are different, an efficient firm produces more output, faces lower prices and earns more profits.

(2) In heterogeneous duopolies, if production costs are the same, a quantitysetter produces more outputs, faces lower prices and earns more profits.

(3) In heterogeneous duopolies, if production costs are different, an efficient firm produces more output, faces lower prices and earns more

profits regardless of whether it is a quantity-setter or a price-setter. If a pricesetter is efficient, it may earn more profits than a quantity-setter even though it produces less output and sets higher prices.

(4) $\pi_x^{CC} > \pi_x^{CB} > \pi_x^{BB} > \pi_x^{BC}$ and $\pi_y^{CC} > \pi_y^{BC} > \pi_y^{BB} > \pi_y^{CB}$ if $x^{BC} > 0$ and $y^{CB} > 0$.

(5) If cost difference is relatively large, an efficient firm makes more profits regardless of its strategy. If cost difference is relatively small, a quantity-setter makes more profits regardless of whether it is efficient.

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